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NOTE

A CHARACTERIZATION OF COMPETITION GRAPHS

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Characterizations of competition graphs for arbitrary and acyclic directed graphs are presented.

Let D be a directed graph having no multiple edges. The competition graph of D is an undirected graph G on the same node set as D and having an undirected edge $\{v_i, v_j\}$ if and only if there exists a third node v_k such that (v_i, v_k) and (v_j, v_k) are directed edges in the edge set of D. Competition graphs of acyclic digraphs were employed by Cohen [1,2,3] to study 'food web' models in ecology. The various animal species in a geographic area were represented by the node set with a directed edge from node v_i to node v_j if species i 'preyed' upon species j. Cohen observed that most food web models tend to be acyclic and justified the restriction to this class of digraphs. The competition graph of such a food web model then exhibits, by undirected edges, those species which compete for food. These graphs have also been studied by Roberts [9,10] and more recently by Opsut [8] who demonstrated that the problem of determining whether or not an arbitrary graph is the competition graph of some acyclic digraph is NP-complete.

Roberts has shown that any undirected graph G can be a competition graph of an acyclic digraph if a sufficient number of isolated nodes are appended to G. Thus we may reject approaches which attempt to characterize competition graphs by 'forbidden' subgraphs. Our first result characterizes competition graphs of arbitrary digraphs (cycles and loops allowed). In the following $V(G)$ is the node set of G, $E(G)$ is the edge set and $\theta_1(G)$ is the minimal number of complete subgraphs which cover the edges.

Theorem 1. *G is the competition graph of an arbitrary digraph D if and only if $\theta_1(G) \le n$.*

Proof. With G the competition graph of D define, for $1 \le i \le n$, C_i as the subgraph of G induced by $\{v_j \mid (v_j, v_i) \in E(D)\}$. Clearly each C_i is a complete subgraph of G and every edge of G is in some C_i. Thus $\theta_1(G) \le n$. Now assume $\theta_1(G) = k \le n$ and let $C_1, C_2, \ldots, C_k$ be an edge cover of G by complete subgraphs. Construct D with $V(D) = \{v_1, v_2, \ldots, v_n\}$ and $(v_i v_j) \in E(D)$ if and only if $v_i \in V(C_j)$. G is then the competition graph for D. □

Notice that D, constructed as in the proof, may contain loops. For a characterization which does not allow loops see Roberts and Steif [11]. Other characterizations are given by Lundgren and Maybee [6].

We now consider the special case of characterizing competition graphs of acyclic digraphs. We shall need the following result [5].

Lemma. *D is an acyclic digraph if and only if its nodes can be labeled so that*

$$(v_i, v_j) \in E(D) \quad \textit{implies} \quad i < j.$$

Theorem 2. *The following statements are equivalent for an undirected graph G on n nodes:*

(a) *G is the competition graph of some acyclic directed graph D.*

(b) *G has a vertex labeling $v_1, v_2, \ldots, v_n$ so that there are complete subgraphs $C_1, C_2, \ldots, C_n$ such that*

(i) *$v_i \in V(C_j)$ implies $i < j$, and*

(ii) *the C_i's form an edge cover of G.*

(c) *G has complete subgraphs $C_1', C_2', \ldots, C_{n-2}'$ which form an edge cover of G such that $|C_1' \cup C_2' \cup \cdots \cup C_j'| \le j+1$ for $1 \le j \le n-2$.*

Proof. (a) ⇒ (b). Choose a vertex labeling $v_1, v_2, \ldots, v_n$ of D as prescribed by the lemma. As before, for $1 \le j \le n$, define C_j as the subgraph of G induced by $\{v_i \mid (v_i, v_j) \in E(D)\}$. Clearly C_j is complete and (i) and (ii) are satisfied.

(b) ⇒ (c). The first condition of (b) implies C_1 and C_2 have no edges. Thus we may define, for $1 \le i \le n-2$, $C_i' \equiv C_{i+2}$. For $1 \le j \le n-2$, if $v_i \in C_1' \cup C_2' \cup \cdots \cup C_j'$, then $i \le j+1$ by (i).

(c) ⇒ (a). Identify the nodes of G in the following way. Label as v_n a node not in $\bigcup_{i=1}^{n-2} C_i'$, as v_{n-1} a different node not in $\bigcup_{i=1}^{n-3} C_i'$, etc. Finally, arbitrarily label the remaining two nodes as v_2 and v_1. Let D be the directed graph on this set of nodes with

$$E(D) = \{(v_i, v_j) \mid v_i \in C_{j-2}'\}.$$

G is easily seen to be the competition graph of D. Furthermore, $(v_i, v_j) \in E(D)$ implies $i \le j-1$. Thus D is acyclic by the lemma. □

It is interesting to note that characterization (b) in Theorem 2 is similar to that of Mukhopadhyay [7] for squares of graphs and Acharya and Vartak (as reported by Escalante, Montejano and Rojano [4]) for neighborhood graphs.

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