(1,2)-Step Competition Graph Examples

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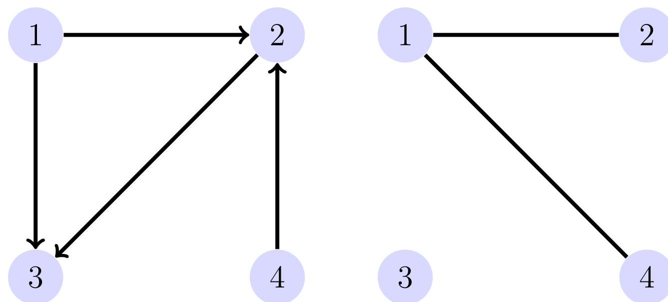
1 Introduction

This document is a web of examples constructed from the definitions found in *The (1,2)-step competition number of a graph* by Factor et al. We present an example of each definition, lemma, etc... after the actual statement of it; our examples range from images to descriptions, where relevant.

2 Definitions

Note: before continuing with the definitions, we let D represent an arbitrary digraph, $V(D)$ be its vertex set, and $A(D)$ be its arc set. Furthermore, we let G represent an arbitrary graph, $V(G)$ be its vertex set, and $E(G)$ be its edge set. We let K_n denote a complete graph (digraph) of n vertices; we will specify whether we are using a graph or digraph. These objects are what we will work with unless otherwise specified.

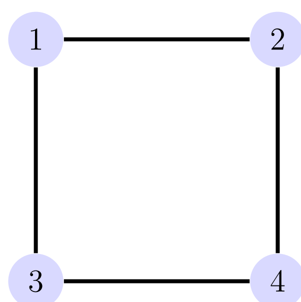
Definition 2.1. Competition graph: the *competition graph of* D, $C(D)$ is the graph with vertex set $V(D)$ such that $\{u, v\}$ is an edge if vertices u and v have a common prey in D. See the example below.

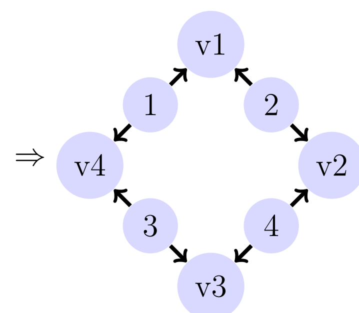


Example digraph (left) and its corresponding competition graph (right).

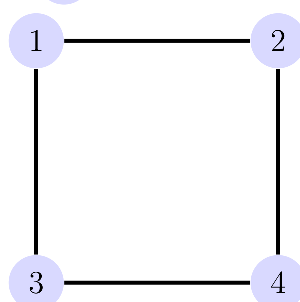
Definition 2.2. Competition number: the *competition number* $k(G)$ of a graph G is the smallest nonnegative integer k so that G, together with k isolated vertices, is the competition graph of some acyclic digraph. See example below.

Given:



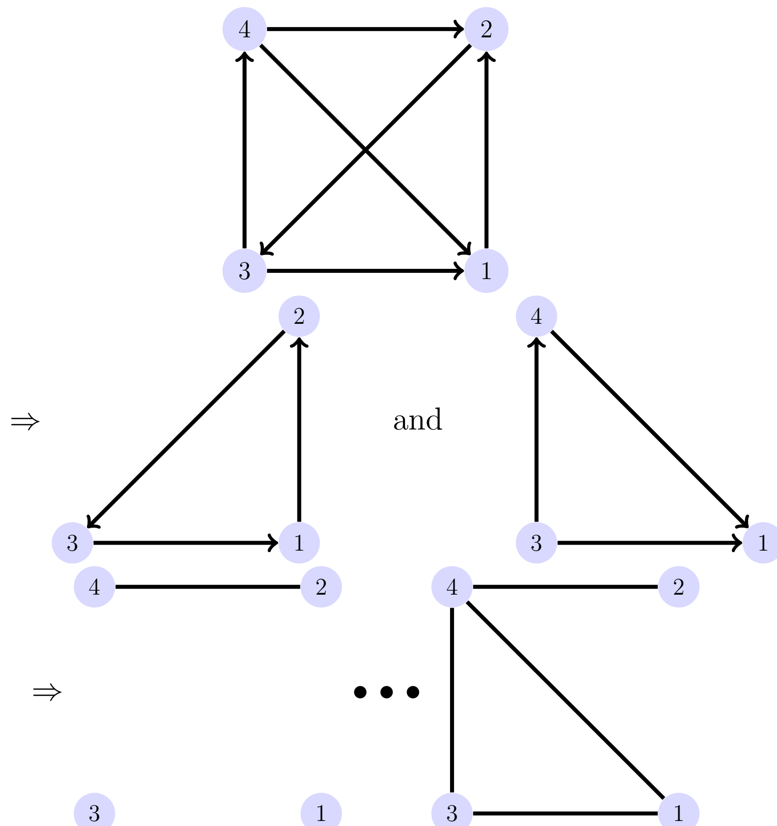


Yields:



The figures above show an example of competition number. In this case, the competition number would be four since four vertices needed to be added.

Definition 2.3. (1,2)-step competition graphs: A (1,2)-step competition graph, or $C_{1,2}$(D), is a graph with a vertex set V(D) in which distinct vertices x and y will make an edge $\{x,y\}$ only when some vertex $(z) \in V(D)$, $d_{D-y}(x,y) \leq 2$ and $d_D(y,z) = 1$ or $d_D - x(y,z) \leq 2$ and d_D (x,z) =1. See the example below for the process of making a (1,2)-step competition graph. For convenience, however, we only show the first few steps to demonstrate the overall process.



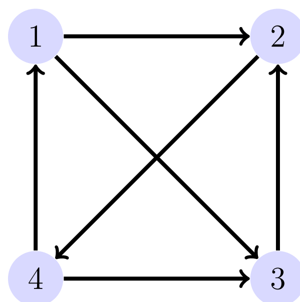
We begin with an example digraph. Then, we select two vertices to investigate—see whether or not they'll have an edge in the $(1,2)$-step competition graph. We begin by selecting vertices 4 and 2. Then, we examine the digraphs $D-4$ and $D-2$, the images are displayed in that order. For our graphs $D-4$ and $D-2$, we must select a vertex, call it z, such that $d_{D-4}(2,z) \leq 2$ and $d_{D-2}(4,z) = 1$ or $d_{D-4}(2,z) \leq 2$ and $d_{D-2}(4,z) = 1$. We must enumerate the possibilities of z until we find that at least one of them works, or none of them work. As we can see, the choice of $z = 1$ works (the reader is left to verify this value works). Thus, we know our $(1,2)$-step competition graph has an edge between vertices 4 and 2. We continue this process for all vertex combinations of x, y and all possibilities for z until we have exhausted them. Therefore, we end with the last graph.

Definition 2.4. (i,m)-step Competition Graph: the (1,2)-step competition graph can be generalized as a (i,m)-step competition graph if for some $z \in V(T) - \{x,y\}$, $d_{T-y}(x,z) \leq i$ and $d_{T-x}(y,z) \leq m$ or $d_{T-x}(y,z) \leq i$ and $d_{T-y}(x,z) \leq m$.

This follows the same process as the (1,2)-step competition graph, but looks for values that are equal to whichever selection of i, m has been made.

Definition 2.5. (i, m)-step competition number: the (i, m)-*step competition number* of a graph G is the minimum number k such that G with k isolated vertices is the (i, m)-step competition graph of an acyclic digraph. Please, see the explanation below.
The (i, m)-step competition number is the (i, m)-step competition graph's version of a competition number of a graph G. That is, we determine it for a specific graph by determining what i, m are and then we begin adding isolated vertices to the vertex set of G. Then, we redefine the arc set of G to create a digraph such that our original G is the (i, m)-step competition graph of G.

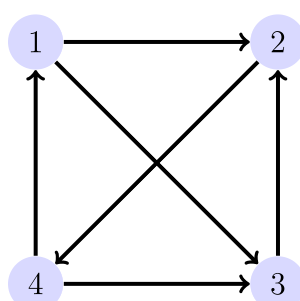
Definition 2.6. Out-Neighborhood of vertex x: the set of all vertices to which x has an arc in D, denoted $N_D^+(x)$



The out-neighborhoods for the vertices in the above digraph are below.

$$N_D^+(1) = \{2, 3\}$$
$$N_D^+(2) = \{4\}$$
$$N_D^+(3) = \{2\}$$
$$N_D^+(4) = \{1, 3\}$$

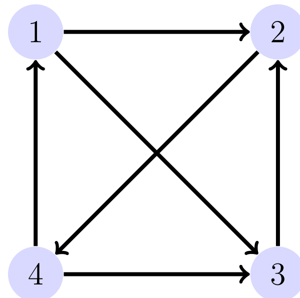
Definition 2.7. In-Neighborhood of a vertex x: the set of all vertices from which x has an arc in D, denoted $N_D^-(x)$



The in-neighborhoods for the vertices in the above digraph are below.

$$N_D^-(1) = \{4\}$$
$$N_D^-(2) = \{1, 3\}$$
$$N_D^-(3) = \{1, 4\}$$
$$N_D^-(4) = \{2\}$$

Definition 2.8. Outdegree of a vertex x: the cardinality of the out-neighborhood of x, denoted $\deg^+_D(x)$



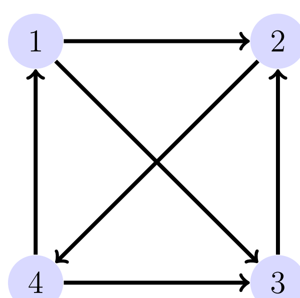
The out-neighborhoods for the vertices in the above digraph are below.

$$N^+_D(1) = \{2, 3\}$$
$$N^+_D(2) = \{4\}$$
$$N^+_D(3) = \{2\}$$
$$N^+_D(4) = \{1, 3\}$$

The outdegrees for the vertices in the above digraph D are below.

$$\deg^+_D(1) = |N^+_D(1)| = 2$$
$$\deg^+_D(2) = |N^+_D(2)| = 1$$
$$\deg^+_D(3) = |N^+_D(3)| = 1$$
$$\deg^+_D(4) = |N^+_D(4)| = 2$$

Definition 2.9. Indegree of a vertex x: the cardinality of the in-neighborhood of x, denoted $\deg^-_D(x)$



The in-neighborhoods for the vertices in the above digraph are below.

$$N^-_D(1) = \{4\}$$
$$N^-_D(2) = \{1, 3\}$$
$$N^-_D(3) = \{1, 4\}$$
$$N^-_D(4) = \{2\}$$

The indegrees for the vertices in the above digraph D are below.

$$\begin{aligned}
\deg_D^-(1) &= |N_D^-(1)| = 1 \\
\deg_D^-(2) &= |N_D^-(2)| = 2 \\
\deg_D^-(3) &= |N_D^-(3)| = 2 \\
\deg_D^-(4) &= |N_D^-(4)| = 1
\end{aligned}$$

3 Lemmas

Lemma 3.1. For all positive integers $n \geq 5,\ k_{(1,2)}(C_n) > 1$ See interpretation below.
The lemma above is saying that for all positive integers, n will be greater than or equal to a (1,2)-step competition number in a cyclic graph with n vertices that is less than one.

4 Propositions

Proposition 4.1. Let i, m be positive integers. For any graph G, there exists a nonnegative integer k such that G with k isolated vertices is the (i, m)-step competition graph of an acyclic digraph. See our interpretation below.
This theorem is a proof of existence for the (i, m)-step competition number for any graph G. Without this theorem, we would not know when the (i, m)-step competition number is actually relevant for a graph.

Proposition 4.2. Let i, m be positive integers. If a graph G has no isolated vertices, then $k_{(i,m)}(G) \geq 1$.See our interpretation below.
This theorem is showing that an acyclic digraph will have a vertex with no out-neighbor so the vertex in the (i, m)-step competition graph will be isolated.

5 Theorems

Theorem 5.1. Let i, m be positive integers. For $n \geq 2$, we have $k_{(i,m)}(K_n) = 1$. For an explanation, please see below.
This theorem is telling us that for any complete graph with vertices two or more, we know its (i, m)-step competition number is equal to one. That is, we can construct a digraph such that its (i, m)-step competition graph is our original complete graph.

Theorem 5.2. For all positive integers $n > 1, k_{(1,2)}(P_n) = 1$. See explanation below.
This theorem is saying that for any complete graph with more than one vertex, we know that the (1,2)- step competition number with a path of n vertices is equal to one.

Theorem 5.3. Let $n \geq 3$ be a positive integer. Then $k_{(1,2)}(C_n) = 1$ when $n = 3, 4$. For $n \geq 5, k_{(1,2)}(C_n) = 2$. For an explanation, please see below.
This theorem tells us that for any cycle graph consisting of 3 or more vertices, we know its $(1, 2)$-step competition number is simply 1 when it has exactly 3 or 4 vertices, and when the cycle graph has 5 or more vertices, then its $(1, 2)$-step competition number is simply 2.

Theorem 5.4. Let n be a positive integer. Then $k_{(1,2)}(K_{1,n}) = 1$. See the explanation below.
This theorem tells us that for any $(1, n)$ complete graph, its $(1, 2)$-step competition number is one.