Competition and (1,2)-Step Competition Numbers in 4-Cycle Graph Variations

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Abstract In this paper, I discuss the competition and (1,2)-step competition number in 4-cycle graphs. These include looking at unions of 4-cycle graphs and graphs with pendants. The competition and (1,2)-step competition numbers are given for these graphs.
Keywords: competition number; (1,2)-step competition number; 4-cycle graphs

1 Introduction

Competition graphs were introduced by Cohen so that food webs could be better analyzed [1]. This is done by representing animal species as nodes and using arcs to show the predator/ prey relationship [2]. The arc (x, y) denotes species x preying on species y. When looking at a digraph, D, the *competition graph of* $D, C(D)$, is the graph with a vertex set $V(D)$ in which u, v is an edge only when vertices u and v share a common prey in D.
The idea of competition graphs were expanded when Factor and Merz introduced the (1,2)-step competition graph [2]. Let D be a digraph. If x and y are vertices in D, then $d_D(x, y)$ is going to be the number of arcs on

the shortest directed path from x to y in D. This means that we are looking for the shortest distance between x and y in D. The (1,2)- *step competition graph* of D, denoted $C_{(1,2)(D)}$, is a graph with vertex set $V(D)$ so that, for vertices x and y, x, y is an edge in $C_{(1,2)(D)}$ if and only if for some vertex $z \epsilon V(D), d_{(D-y)}(x,z) \leq 2$ and $d_D(y,z) = 1$ or $d_{(D-x)}(y,z) \leq 2$ and $d_D(x,z) = 1$.
The competition graph can be generalized to be an (i,j)-step competition graph. This was introduced by Hefner (Factor) [3]. Let x, y be an edge in the (i,j)-step competition graph, denoted $C_{(i,j)}(D)$, if for some $z \epsilon V(T) - x, y, d_{(T-y)}(x,z) \leq i$ and $d_{(T-x)}(y,z) \leq j$ or $d_{(T-x)}(y,z) \leq i$ and $d_{(T-y)}(x,z) \leq j$.
Roberts introduced the concept of a competition number, $\gamma(D)$ [2]. The competition number of graph G is defined to be the smallest nonnegative integer k so that G, together with k isolated vertices, is the competition graph of some acyclic digraph. The competition number can be expanded to the (1,2)-step competition number which was introduced by Factor and Merz [2]. This is the same concept, expect the purpose is to get the (1,2)-step competition graph of some acyclic digraph.
In this paper, the indegree and outdegree of a vertex x in digraph D will be denoted as $d^-(x)$ and $d^+(x)$. [2].
Previous research was done on competition and (1,2)-step competition numbers by Factor, Merz, and Sano [2]. In one of their papers, they brought up the question on if other graphs besides 4-cycle (C_4) graphs held the same results they got in which the competition number was greater than the (1,2)-step competition number. This paper exhibits my work in which I attempted to come up with an answer to that question.

2 Unions of C_4 Graphs

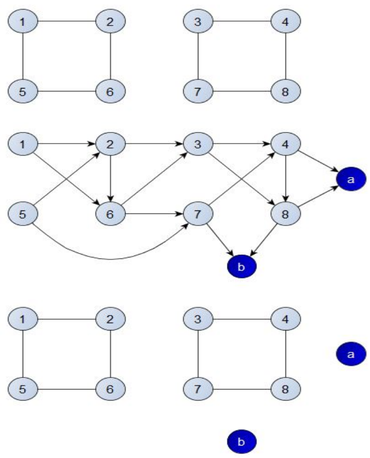

Figure 1: Figure 1: $C_4 \cup C_4$ $\gamma(C_4 \cup C_4) = 2$

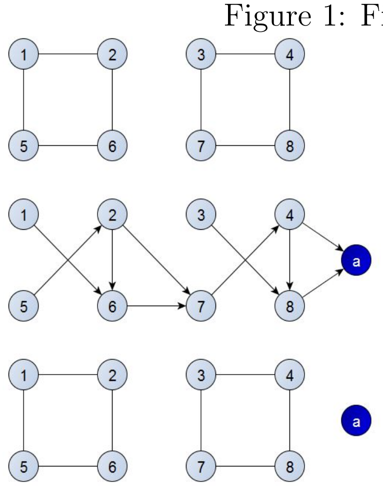

Figure 2: Figure 2:$C_4 \cup C_4$ $\gamma_{(1,2)}(C_4 \cup C_4) = 1$

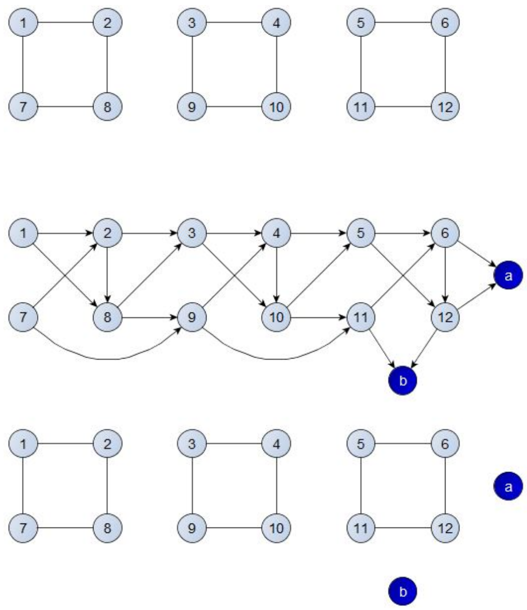

Figure 3: Figure 3: $C_4 \cup C_4 \cup C_4$ $\gamma(C_4 \cup C_4 \cup C_4) = 2$

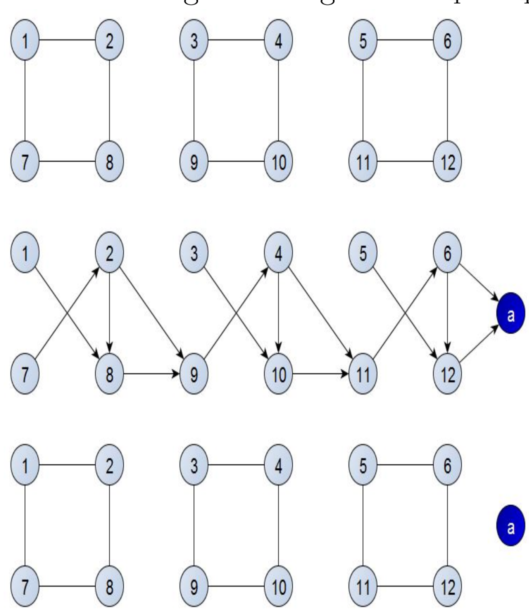

Figure 4: Figure 4: $C_4 \cup C_4 \cup C_4$ $\gamma_{(1,2)}(C_4 \cup C_4 \cup C_4) = 1$

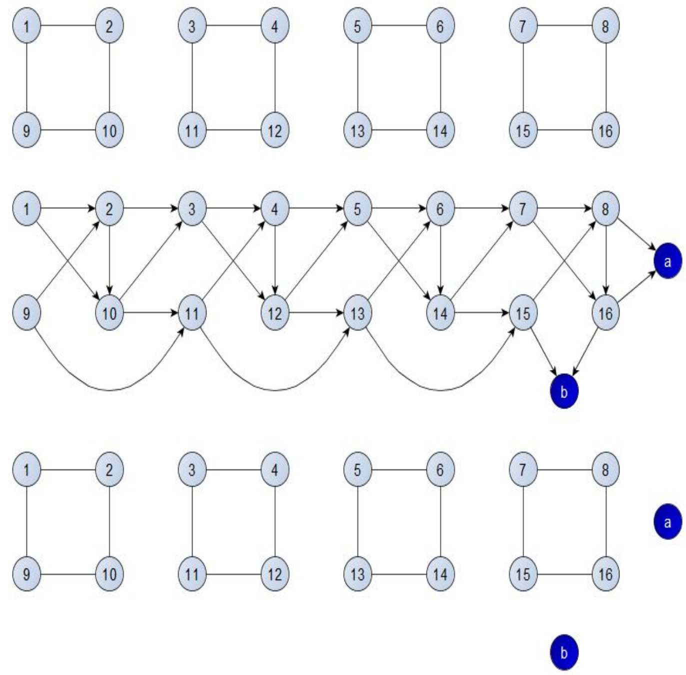

Figure 5: Figure 5: $C_4 \cup C_4 \cup C_4 \cup C_4$ $\gamma(C_4 \cup C_4 \cup C_4 \cup C_4) = 2$

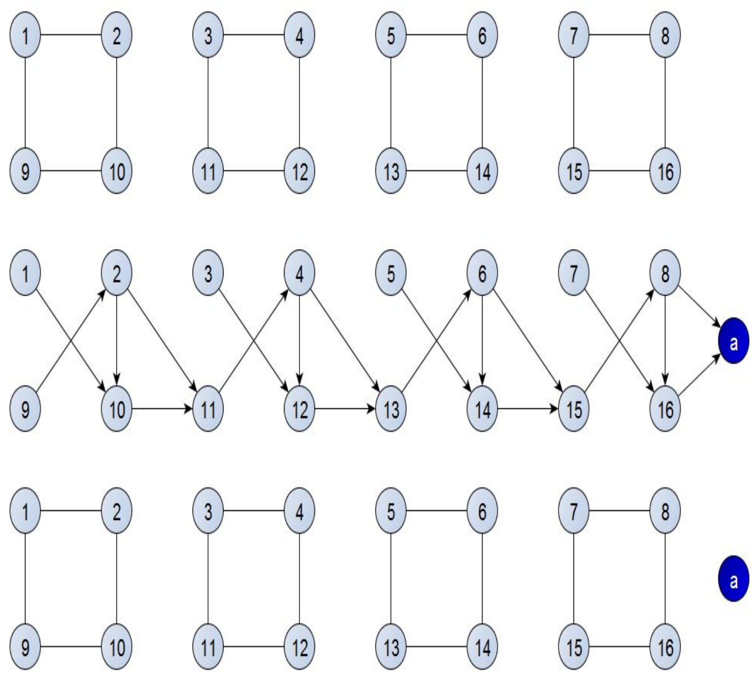

Figure 6: Figure 6: $C_4 \cup C_4 \cup C_4 \cup C_4$ $\gamma_{(1,2)}(C_4 \cup C_4 \cup C_4 \cup C_4) = 1$

Lemma 2.1. *If* $G = \cup_{i=1}^{k} C_4, k \geq 1, \forall x \in VC_4.d^-(x) \leq 2$ *for both y(D) and* $y_{(1,2)}(D)$*.*

Proof. Looking at Figures 1 and 2, it can be seen that there are no triangles and each vertex has no more than three vertices competing for it. If more than three vertices compete for the same vertex, there would be a triangle in G. Therefore, $d^-(x) \leq 2$.

Lemma 2.2. *If* $G = \cup_{i=1}^{k} C_4, k \geq 1, \forall x \in VC_4.d^+(x) \geq 2$ *for y(D).*

Proof. By looking at Figure 1, it can be seen that every vertex in D is adjacent to two other vertices. As stated in lemma 2.1, no three vertices compete since there is no triangle in $C_4 \cup C_4$. Therefore, each vertex must have an outgoing arc for each edge. Thus, $d^+(x) \geq 2$.

Proposition 2.3. *Given* $G = \cup_{i=1}^{k} C_4, k \geq 1, \gamma(D) = 2$.

Proof. As seen in figures 1, 3, 5, $\gamma(D) = 2$. This would remain true for $\cup_{i=1}^{k}$ $C_4, k \geq 1$ because of lemmas 2.1 and 2.2.

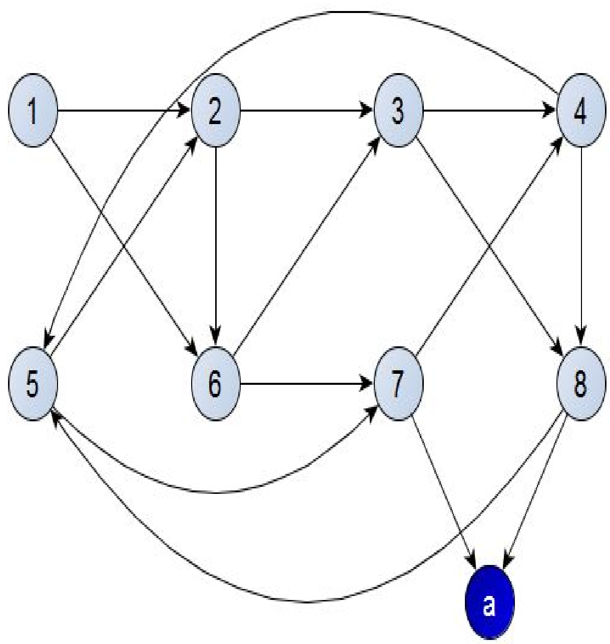

Figure 7: Figure 7: A visual of $\gamma(C_4 \cup C_4) = 1$.

Figure 7 shows why $\gamma(C_4 \cup C_4)$ could not be one. The digraph shown in Figure 7 is a cycle. The definition of a competition number states that the digraph must be acyclic, therefore this would not work for $C_4 \cup C_4$ or for any graph in which $G = \cup_{i=1}^{k} C_4, k \geq 1$.

Proposition 2.4. *Given* $G = \cup_{i=1}^{k} C_4, k \geq 1, \gamma_{(1,2)}(D) = 1$.

Proof. As seen in Figures 2, 4, and 6, $\gamma_{(1,2)}(D) = 1$. This would remain true for $\cup_{i=1}^{k} C_4, k \geq 1$ because of lemmas 2.1 and 2.2.

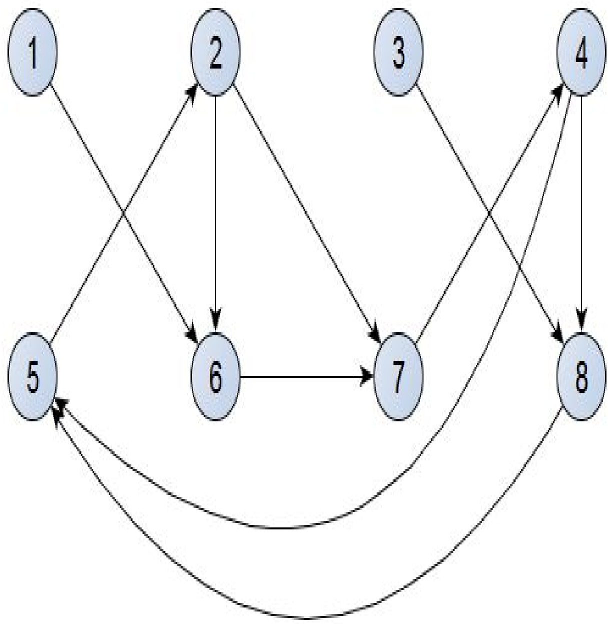

Figure 8: Figure 8: A visual of $\gamma_{(1,2)}(C_4 \cup C_4) = 0$.

Figure 8 shows why $\gamma_{(1,2)}(C_4 \cup C_4)$ could not be zero. The digraph shown in Figure 8 is a cycle. Even though this wouldnt create any additional edges in $C_{(1,2)}(C_4 \cup C_4)$, the definition of a (1,2)-step competition number states that the digraph must be acyclic. This would not work for any graph in which $G = \cup_{i=1}^{k} C_4, k \geq 1$.

Theorem 2.5. *Given* $G = \cup_{i=1}^{k} C_4, k \geq 1, \gamma(D) > \gamma_{(1,2)}(D)$.

Proof. Due to propositions 2.3 and 2.4, $\gamma(D) > \gamma_{(1,2)}(D)$. This is true because $2 > 1$.

3 C_4 Graphs with Pendant

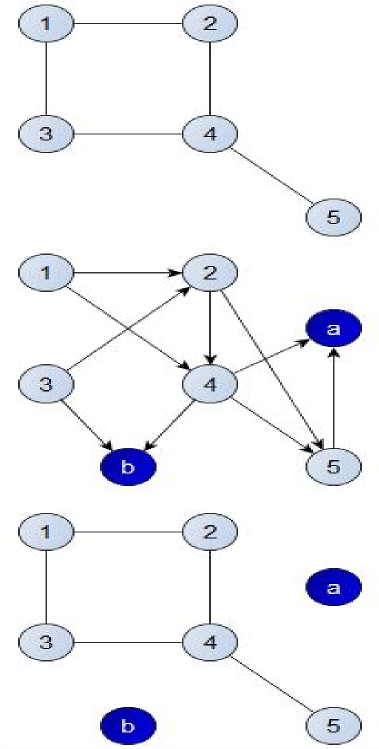

Figure 9: Figure 9: $D = C_4$ with pendant. $\gamma(D) = 2$.

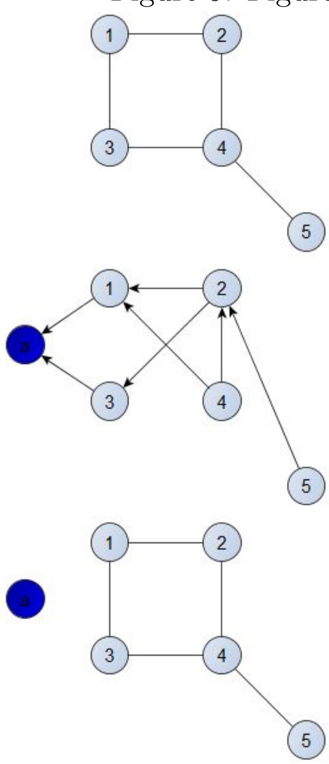

Figure 10: Figure 10: $D = C_4$ with pendant. $\gamma_{(1,2)} = 1$.

Lemma 3.1. *If* $G = C_4 with l pendants \geq 1, \forall x \in VC_4, d^-(x) \leq 2$ *for both* $\gamma(D)$ *and* $\gamma_{(1,2)}(D)$.

Proof. Looking at Figures 9 and 10, it can be seen that there are no triangles and each vertex has no more than three vertices competing for it. If more than three vertices compete for the same vertex, there would be a triangle in G. Therefore, $d^-(x) \leq 2$.

Lemma 3.2. *If* $G = C_4 with l pendants \geq 1, \forall x \in VC_4, d^+(x) \geq 3$ *for* $\gamma(D)$.

Proof. By looking at Figure 9, it can be seen that every vertex in D is adjacent to at least two other vertices. Vertex 4, however, is adjacent with three vertices. As stated in lemma 3.1, no three vertices compete since there is no triangle in C_4. Therefore, almost every vertex must have an outgoing arc for each edge. Thus, $d^+(x) \geq 2$. The only exception is for $d^+(4) = 3$.

Proposition 3.3. *Given* $G = C_4$ *with 1 pendant,* $\gamma(G) = 2$.

Proof. As seen in Figure 9, $\gamma(D) = 2$.

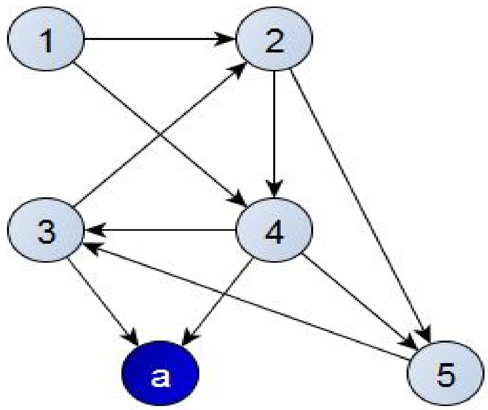

Figure 11: Figure 11: G = C_4 with one pendant, a visual of $\gamma(G) = 1$.

Figure 11 shows why $\gamma(G)$ (with $G = C_4$ with pendant) could not be one. The digraph shown in Figure 11 is a cycle. The definition of a competition number states that the digraph must be acyclic, therefore this would not work.

Proposition 3.4. *Given $G = C_4$ with 1 pendant,* $\gamma_{(1,2)}(G) = 1$.

Proof. As seen in Figure 10, $\gamma_{(1,2)}(D) = 1$.

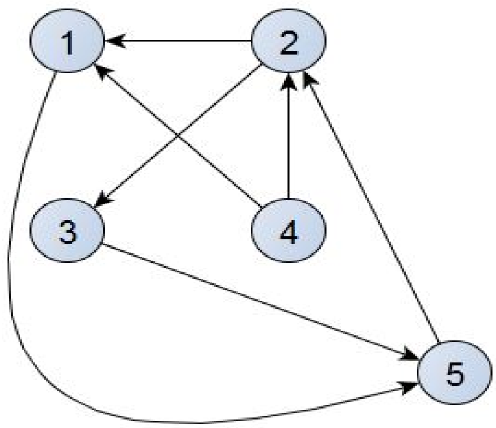

Figure 12: Figure 12: G = C_4 with one pendant, a visual of $\gamma_{(1,2)}(G) = 0$.

Figure 12 shows why $\gamma_{(1,2)}(G)$ could not be zero. The digraph shown in Figure 12 is a cycle. Even though this wouldnt create any additional edges in $C_{(1,2)}(G)$, the definition of a (1,2)-step competition number states that the digraph must be acyclic. This would not work then for C_4 with one pendant.

Theorem 3.5. *Given $G = C_4$ with 1 pendant, $\gamma(D) > \gamma_{(1,2)}(D)$.*

Proof. Due to propositions 3.3 and 3.4, $\gamma(D) > \gamma_{(1,2)}(D)$. This is true because $2 > 1$.

4 Concluding Remarks

This paper went through looking at different variations of C_4 graphs in order to partially answer a previous research question. The results show that there are other graphs besides regular C4 graphs in which the competition number is greater than the (1,2)-step competition number. There are still remaining questions for further study, though.

- Are there other extensions of C4 graphs where $\gamma(D) > \gamma_{(1,2)}(D)$?
- Are there other basic graph structures besides C_4 graphs where this pattern remains true?
- Is there still a pattern when further extensions of pendants are used?

- What patterns exist when taking unions of C_4 graphs and adding an edge between them?

5 References

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